ON METRIC ALGEBRAS

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
June, 1952

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CHAPTER I

INTRODUCTION

Regarded operationally, modern mathematics may be divided into three parts. The first of these is called algebra, and deals with finitary operations. The second may be referred to loosely as topology, and deals with infinite processes and operations. The third includes those fields of mathematics in which both finitary and infinite operations are considered, and is called analysis. This view is essentially that of Solomon Lefschetz (19)[1]. Of the fields of modern analysis, the one most closely related to classical mathematics is probably topological algebra (that is, the study of systems which bear both algebraic and topological structures). We will agree for the present to include under the heading topology the study of geometric properties of metric spaces.

It is the purpose of this dissertation to derive some results in topological algebra by metric methods.

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[1] Numbers in parenthesis indicate references to the bibliography at the end of the dissertation.

The dissertation is divided into two main parts. In the first part, the major objective is to extend to very general types of metric algebras the completing process already used in many specific cases (22) (23). This result will then be applied to determine a characterization among normed commutative groups of subgroups of the additive groups of Banach spaces.

The second part is devoted to a very special kind of metric algebra closely related to various previously studied lattice ordered algebraic structures. References to these relations will be given at the beginning of the second part. The program is essentially to consider a certain class of metric lattices bearing a third operation, to show that a subclass of these is the class of operator lattices of a certain type, and finally, to show that a subclass of this class of operator lattices coincides algebraically with a class of semigroups of prime importance; namely, those semigroups whose arithmetic is a straightforward generalization of the multiplicative part of elementary number theory.

CHAPTER II

PRELIMINARY CONCEPTS

In this chapter will be introduced some notions necessary for later arguments.

1. Sets, functions, and mappings.

A set S is an aggregate of objects, called elements or points, together with some rule for deciding whether or not a given element belongs to the set. A function, or mapping, is a correspondence between elements of a set, or between elements of two different sets. That part of a set for which the correspondence is defined is the domain of the function; that set into which a domain is mapped by a function is called the range of the function. The correspondence need not be one-to-one, nor need all the elements in the range actually be used in the correspondence. A function takes its arguments from the domain, and its values in the range. The function assigns to each set of n elements in the domain a value or set of values in the range, depending on whether the function is single or multiple valued. In the case that n is 1 and the function is single-valued, the mapping terminology particularly applies.

In this case an element in the domain has an image under the mapping, and an element in the range, if it is used in the correspondence, has one or more originals or counter images. The inverse of such a mapping is the correspondence defined in the reverse fashion, or a function which takes images as arguments, and counter images as values. We shall restrict the notion of mapping to single valued functions in this paper.

A function will be given a letter label, such as F, g, or α. F(a) means the image of the element 'a' under the function F. If several mappings operate on the same set, however, we shall find it convenient to use the operator notation for a mapping, in which $a\alpha$ denotes the image of the element 'a' under the mapping α. This facilitates the writing of repeated mappings, as in $(a\alpha)\beta$. A product $\alpha\beta$ of two mappings on a set S is the mapping defined by $x(\alpha\beta) = (x\alpha)\beta$ for all elements x in S.

2. Algebras.

By an algebra, we mean a set A of elements, together with a number of single valued functions taking on values in the same set A. We shall immediately restrict ourselves to algebras possessing only a single function of two arguments whose domain is the set of all pairs of elements from A. Such a function is called a binary operation or composition, and is denoted by writing the two arguments with the symbol for the operation placed between them, as a+b, or merely by juxtaposition, as ab. The set A with this operation is called a groupoid. A given set G may form a groupoid under more than one operation; hence, to be completely clear, the groupoid should be designated as G(ab), where a and b are elements of the set G, and the binary operation is indicated by juxtaposition. However, when the context makes the operation clear, we shall refer to the groupoid merely by its set label.

From this simple beginning, it is possible to arrive at familiar algebraic systems by requiring that additional postulates be satisfied (7). The groupoid property is more succinctly stated as the property of operational closure. If we require that the operation ab be associative, that is, a(bc) = (ab)c for any three elements in the algebra, the resulting system is called a semigroup. If, for any two elements a and b of G, there exists uniquely an x, also in G, such that ax = b, and a unique y such that ya = b, the groupoid G is said to be a quasigroup, and the property is called the quasigroup property. Briefly, it is the property that equations have solutions on both sides. Note that this requires that an associative quasigroup G have an identity, or an element e such that ae = ea = a for all a in G, and that this identity be unique. This, in turn, requires that each element have an inverse, i. e., an element a^{-1} such that $a^{-1}a = aa^{-1} = e$.

One set of requirements on a groupoid in order that it be a group is that the operation be associative, that there exist a unique identity, and that each element have an inverse. Thus we can "factor" a group into a semigroup and a quasigroup in this sense: a quasigroup which is a semigroup is a group.

A subset of a groupoid which is itself a groupoid under the same operation is called a subgroupoid of the original one. This, of course, means that the result of combining any two elements in the subset under the given operation is an element in the subset. If, moreover, the result of combining any element of the set with an element of the subset is an element of the subset, the subset is an ideal of the original groupoid. Clearly, an ideal is a subgroupoid.

If a subgroupoid is of a special nature, it is referred to by the appropriate name indicating this nature. For example, if A is a subset of the groupoid B which forms a group under the same operation, it is called a subgroup of B regardless of whether or not B is a group.

The operation in a groupoid G will ordinarily be called multiplication for convenience. Multiplying all elements of G on the right by a single element 'a' maps G into itself. Such a mapping is called the right translation of G by a, and is denoted by R_a. A left translation, L_a, is defined similarly. If G is commutative (that is, ab = ba for all a and b in G), these mappings coincide, and the mapping is simply a translation, which we shall call R_a.

A homomorphism of one algebra into another is a many-to-one mapping which preserves the operation. More precisely, if f is a mapping of G(ab) into $H(x\Delta y)$, then f is a homomorphism if $f(ab) = f(a) \Delta f(b)$. If a homomorphism maps an algebra into itself, it is an endomorphism. If a homomorphism is one-to-one, or biuniform, it is an isomorphism. Such a mapping need not exhaust the image set. If it does, the isomorphism is said to be onto. Otherwise, the isomorphism is into, and we have embedded one algebra isomorphically in another. An automorphism is an endomorphism which is an isomorphism onto.

A partially ordered set P is a set of elements in which a binary relation $\leq$ (precedes) is defined, which relation is

(P1) reflexive: $x \leq x$ for all x in P;

(P2) antisymmetric: $x \leq y$ and $y \leq x$ imply $x = y$;

(P3) transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$ for all x, y, and z in P.

$x \geq y$ (x follows y) is defined to mean $y \leq x$. Note that not all pairs need be comparable by $\leq$ in a partially ordered set. If all pairs are ordered by $\leq$ (as in the case of the real numbers), the set is said to be *simply*, or *linearly*, ordered. $x < y$ (x properly precedes y) means $x \leq y$ and $x \neq y$.

An *upper bound* for a subset of P is an element u such that $u \geq x$ for all x in the subset; a lower bound is defined dually. A *least upper bound* (l.u.b.) is an upper bound which precedes every other upper bound (and dually for a greatest lower bound, g.l.b.). An arbitrary subset of a partially ordered set may fail to have any one or all of these bounds.

A *lattice* (4) is a partially ordered set P, any two of whose elements have a g.l.b. or meet, and a l.u.b. or join. Then, of course, any finite subset also has a meet and a join. If every subset has a meet and a join, the lattice is said to be *lattice complete*.

A lattice can also be defined as a set L which forms a groupoid under two operations, these operations being $a \wedge b$ and $a \vee b$ (read "a meet b" and "a join b", respectively), satisfying the following postulates:

(L1) Idempotence: $x \wedge x = x \vee x = x$; for all x in L;

(L2) Commutativity: $x \wedge y = y \wedge x$, $x \vee y = y \vee x$; for all x and y in L;

(L3) Associativity: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, and $(x \vee y) \vee z = x \vee (y \vee z)$, for all x, y, and z in L;

(L4) Absorptivity: $x \wedge (x \vee y) = x \vee (x \wedge y) = x$, for all x and y in L.

The two definitions given here can be shown to be equivalent by defining $x \leqslant y$ if and only if $x \wedge y = x$. We shall need the notions inherent in both definitions. The two operations preserve order; that is, if $a \leqslant c$, and $b \leqslant d$, then $a \vee c \leqslant b \vee d$, and $a \wedge c \leqslant b \wedge d$.

In any lattice we have the one-way distributive laws:

(D_1) $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$;

(D_2) $a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c)$;

for all a, b, and c in the lattice. If $\leqslant$ in (D_1) and $\geqslant$ in (D_2) can be replaced with =, the lattice is said to be distributive.

A Hasse diagram for a finite lattice (or any finite partially ordered set) is a diagram in which each element of the lattice is designated by a point in a plane configuration. Then the elements 'a' and 'b' in the lattice are in the relation $a < b$ if and only if the point corresponding to 'a' lies below the point corresponding to 'b', and is connected to it by a segment or sequence of segments.

A sublattice is a subset of a lattice which contains, with any two elements, their meet and join. A lattice which is not distributive must contain one of the following sublattices:

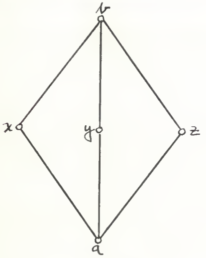

Figure 1

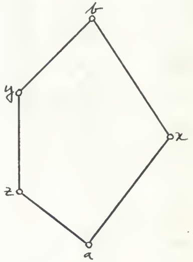

Figure 2

A lattice is relatively complemented if, for $a \leq x \leq b$, there exists an element y in the lattice such that $x \wedge y = a$, and $x \vee y = b$. In neither of the above sublattices are relative complements unique. A necessary and sufficient condition that a lattice be distributive is that relative complements be uniquely determined.

A modular lattice is one in which the following condition is satisfied:

(M) $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.

A distributive lattice is necessarily modular. Every non-modular lattice contains a sublattice isomorphic to the lattice in Figure 2.

A first element 0 for a lattice is an element which precedes all the elements in the lattice. A lattice is atomic, if for $x > 0$, there is an element y with $0 < y \leqslant x$ such that $z \leqslant y$ implies $z = 0$ or $z = y$. The element y is called an atom. Roughly, atomicity means that there are elements of the lattice immediately following the first element. This is not true in the lattice of non-negative real numbers, since there is no first element following the number 0.

A lattice isomorphism is a one-to-one-mapping of one lattice into another which preserves both operations. A lattice endomorphism is a many-to-one mapping of a lattice into itself which preserves both operations.

A real vector space is a commutative group $G(a+b)$ (it is customary to use the additive notation for a commutative group, where $-x$ is the inverse of x, and 0 is the group identity) which is closed under multiplication by real numbers, which multiplication satisfies:

(V1) $(\alpha + \beta)x = \alpha x + \beta x$,

(V2) $\alpha(x + y) = \alpha x + \alpha y$,

(V3) $\alpha(\beta x) = (\alpha\beta)x$,

(V4) $1x = x$,

where α and β are any real numbers, and x and y are any elements of G. A rational vector space (12) is a commutative group which admits multiplication by rationals in the same manner.

3. Equivalence relations.

An equivalence relation is a binary relation between elements of a set which satisfies the following postulates:

(E1) Reflexivity: aRa for all a in the set; or an element must be in the relation with itself;

(E2) Symmetry: aRb implies bRa;

(E3) Transitivity: aRb and bRc imply aRc.

As a simple example, equality among real numbers is an equivalence relation.

The presence of an equivalence relation in a set allows us to partition the set into equivalence classes, grouping together those elements which are in the relation with each other. Oftentimes this enables us to reason on equivalence classes as though they were single elements, the reasoning being valid for all members of the class.

An equivalence relation partitions a set into disjoint classes; for, if there were an element common to two classes, by (E3) all elements of both classes would be in the relation with each other. On the other hand, if a set is partitioned into disjoint classes, we can define an equivalence relation on the set by putting xRy if and only if x and y are in the same class.

4. Gaussian semigroups (8) (16).

Let G(ab) be a commutative semigroup with identity e, in which ax = ay implies x = y (the *cancellation law*). A *unit* in G is an element which has an inverse. For a and b in G, we say that a divides b, written a|b, if there is an element x in G such that ax = b. If a|b and b|a, a and b are called *associate* elements. The relation of associateness is easily seen to be an equivalence relation. A *proper factor* of an element a is an element b such that b|a, and b is neither a unit nor an associate of a. An element is irreducible if it is not a unit and has no proper factors.

A factorization of an element into the product of irreducibles is *essentially unique* if, for any other factorization into irreducibles, the number of factors in each factorization is the same, and each irreducible in the first factorization is associate to an irreducible in the second.

A *Gaussian semigroup* (or *arithmetic*) is a commutative semigroup with identity and cancellation in which every non-unit has an essentially unique factorization into irreducible elements.

Let G(ab) be a Gaussian semigroup. If we partition

G into equivalence classes by the relation of associateness, we get another Gaussian semigroup G'. In G' there are no associate elements, and there is only one unit, the identity element. Hence, factorization into irreducibles in G' is unique. G' is called a *reduced Gaussian semigroup*.

A *greatest common divisor* (g.c.d.) of a and b in the Gaussian semigroup G(ab) is an element d such that $d|a$ and $d|b$, and if $c|a$ and $c|b$, then $c|d$. A *least common multiple* of a and b is an element m such that $a|m$ and $b|m$, and if $a|k$ and $b|k$, then $m|k$. Every pair of elements in a Gaussian semigroup has a g.c.d. and a l.c.m.

5. *Metric spaces*.

A *metric space* M is a set together with a non-negative, single valued, real valued function $\partial(x,y)$ defined on each pair of elements of M, which satisfies the following:

(M1) Vanishing condition: $\partial(a,b) = 0$ if and only if $a = b$;

(M2) Symmetry condition: $\partial(a,b) = \partial(b,a)$ for all pairs;

(M3) Triangle inequality; $\partial(a,b) + \partial(b,c) \geq \partial(a,c)$; for all a, b, and c in M.

The *metric* $\partial(x,y)$ induces a *topology* on the set M in this manner: a point p is in the *closure* of a subset A of M if every sphere S(p,r) (the points x of M such that $\partial(x,p) < r$, where $r > 0$ is the radius of the sphere) has points in common with the set A. It follows that every point of A is in the closure of A, denoted $\bar{A}$, since p is in every such sphere. If the closure of a subset of M is M itself, the subset is said to be *dense* in M. If there are points of A other than p in every sphere, then p is an *accumulation point* (of the subset A or of the space M). In this case, there must be an infinite number of points of A in every sphere about p, since if there were only a finite number, there would be a nearest, and a sphere of radius less than this minimum distance would exclude all points of A other than p itself. From this infinite set, it is possible to choose a sequence of points $(x_n) = x_1, x_2, \ldots,$ such that for any $\epsilon > 0$, there exists an integer N such that for $n > N$, $\partial(x_n, p) < \epsilon$. Such a sequence of points is said to be *convergent*, converging to the *limit* p. This can be briefly written as $\lim_{n \to \infty} x_n = p$. Limits of sequences, if they exist, are unique in metric spaces.

A *Cauchy* (*regular*, *fundamental*) sequence (x_n) is one for which there exists, for any $\epsilon > 0$, an integer N such that for $n, m > N$, $\partial(x_n, x_m) < \epsilon$. Note that this sequence need not have a limit in an arbitrary metric space. If every such sequence has a limit, the metric space is said to be *complete*. The real numbers enjoy this property, although the rationals do not. The

Cantor completion of the rationals is accomplished by taking as elements of a new space equivalence classes of Cauchy sequences of rational numbers. These are defined to be real numbers. The rationals then are isomorphic to a subset of these equivalence classes, namely those that had rational limits.

An *isometry* is a one-to-one mapping of one metric space into another which preserves distance; i.e., the distance between two points in the domain of the mapping is the same as the distance between their images.

We will now consider the generalization to any metric space of the method of completion outlined above. The proof of the following theorem is translated and adapted from Kuratowski (17).

The Hausdorff Theorem: *Any metric space is isometric to a dense subset of a complete metric space*.

Proof. Let M be an arbitrary metric space. We take as elements in a new space C' all Cauchy sequences of elements from M. A point of C' is the sequence (x_n) of points of M. Form equivalence classes of elements of C' under the relation X R Y, where $X = (x_n)$ and $Y = (y_n)$, if and only if $\lim_{n \to \infty} \partial(x_n, y_n) = 0$. This yields a new space C, the space of equivalence classes of C', and the point X in the space C will denote the equivalence class of Cauchy sequences from M which contains (x_n). In C, define $\rho(X, Y)$ to be $\lim_{n \to \infty} \partial(x_n, y_n)$, where (x_n) and

(y_n) are any representatives from the equivalence classes X and Y, respectively. Now, if (x'_n) and (y'_n) are any two different representatives, we have $\partial(x_n, y_n) \leq \partial(x_n, x'_n) + \partial(x'_n, y'_n) + \partial(y'_n, y_n)$, by (M3). Taking the limit as $n \to \infty$ of both sides of this inequality, we have that, if both limits exist, $\lim_{n\to\infty} \partial(x'_n, y'_n) \geq \lim_{n\to\infty} \partial(x_n, y_n)$. Similarly, $\lim_{n\to\infty} \partial(x'_n, y'_n) \leq \lim_{n\to\infty} \partial(x_n, y_n)$, so that if these limits exist, they are equal, and the definition of $\rho(X,Y)$ is independent of representatives. To show that the limit exists, we observe

$$\partial(x_n, y_n) \leq \partial(x_n, x_m) + \partial(x_m, y_m) + \partial(y_m, y_n),$$

or $$\partial(x_n, y_n) - \partial(x_m, y_m) \leq \partial(x_n, x_m) + \partial(y_n, y_m).$$

The right side of the last inequality can be made arbitrarily small, so we have here the condition that a sequence of real numbers be Cauchy. The reals are complete; hence, a limit for this sequence of reals exists. Thus, $\rho(X,Y)$ is defined for all pairs in C, and this definition is independent of representatives.

The vanishing condition and the symmetry condition are obvious from the definition, so it remains to check the triangle inequality. Since $\partial(x_n, y_n) + \partial(y_n, z_n) \geq \partial(x_n, z_n)$, by passing to the limit on n we get that $\rho(X,Y) + \rho(Y,Z) > \rho(X,Z)$. Hence, C is a metric space.

The sequence from M in which every member is the point x is clearly a Cauchy sequence, so we expect to find its equivalence class in C. For two such sequences, (x) and (y), $\rho(X,Y) = \lim_{n\to\infty} \partial(x,y) = \partial(x,y)$. Thus M is isometric to a subset of C under the mapping $x \to \{(x)\}$, where $\{(x)\}$ denotes the equivalence class of which (x) is a member.

It remains to prove that C is complete. Let $X_1, X_2, \ldots$ be any sequence of elements from C, where X_n is the equivalence class of the sequence $x_1^n, x_2^n, \ldots$ of elements from M. We use the superscript n to indicate its relation to X_n. There exists a sequence of positive integers $m_1, m_2, \ldots$ such that

$$(1) \qquad \partial(x_{m_n}^n, x_i^n) < \frac{1}{n}, \text{ for } i > m_n,$$

since each (x_k^n) is a Cauchy sequence. Now suppose that (X_n) is a Cauchy sequence in C. We wish to prove that it has a limit; that is, that there is some Cauchy sequence from M whose equivalence class X is $\lim_{n\to\infty} X_n$.

Consider the sequence class $X = \{x_{m_1}^1, x_{m_2}^2, \ldots\}$, where the m_n are the integers selected above. Select $\epsilon > 0$. The sequence $X_1, X_2, \ldots$ is Cauchy; hence, there exists an integer $q > \frac{1}{\epsilon}$ such that $\rho(X_q, X_{q+k}) < \epsilon$ for all k. From the definition of $\rho(X,Y)$, there must exist a sequence of integers j_k (clearly j depends on k, hence the subscript) such that

$$(2) \qquad \partial(x_i^q, x_i^{q+k}) < \epsilon \text{ for } i > j_k.$$

Then the inequality

$$a(x^q_{m_q}, x^{q+k}_{m_{q+k}}) \leq a(x^q_{m_q}, x^q_i) + a(x^q_i, x^{q+k}_i) + a(x^{q+k}_i, x^{q+k}_{m_{q+k}})$$

implies, with (1) and (2), that for i greater than m_q, j_k, and m_{q+k},

$$(3) \quad a(x^q_{m_q}, x^{q+k}_{m_{q+k}}) \leq \frac{1}{q} + \epsilon + \frac{1}{q+k} < 3\epsilon \text{ for all } k.$$

Hence $X = \{x^1_{m_1}, x^2_{m_2}, \ldots\}$ is a Cauchy sequence class. The inequality

$$a(x^n_i, x^i_{m_i}) \leq a(x^n_i, x^n_{m_n}) + a(x^n_{m_n}, x^i_{m_i})$$

implies that for $i > m_n$, $n > q$, and $i > q$,

$$(4) \qquad a(x^n_i, x^i_{m_i}) < \frac{1}{q} + 6\epsilon < 7\epsilon .$$

Thus, for any $\epsilon > 0$, there exists an integer q such that for $n > q$, the inequality (4) holds, in case i is sufficiently large. As i becomes infinite, we have $\rho(X_n, X) = \lim_{i\to\infty} a(x^n_i, x^i_{n_i}) \leq 7\epsilon$, or $\lim_{n\to\infty} X_n = X$.

The subset of C considered before, namely, that consisting of sequences of iterated elements from M, is dense in C; that is to say, every point of C is in the closure of this subset. To show this, consider any element of C, say $X = \{x_1, x_2, \ldots\}$. It can be written as $\lim_{k\to\infty} X_k$, where $X_k = \{x_k, x_k, \ldots\}$, for $k = 1, 2, \ldots$.

5. *Metric algebras*.

A single valued function defined on a metric space M and taking values in another metric space K (which may be M) is said to be *continuous at a point* p in M provided, for any sequence (x_n) in M with p as a limit, $\lim_{n\to\infty} f(x_n) = f(p)$. If $\delta(a,b)$ is the metric in M and $\rho(c,d)$ the metric in K, this definition can be stated thus: for any $\epsilon > 0$, there exists $\eta > 0$ such that if $\delta(x_n,p) < \eta$, then $\rho(f(x_n),f(p)) < \epsilon$. A function is continuous over a subset of a metric space if it is continuous at each point of the subset.

A function of two arguments in a metric space M taking values in another metric space is said to be *simultaneously continuous* for the arguments p and q if for (x_n) approaching p and (y_n) approaching q, we have $\lim_{n\to\infty} f(x_n,y_n) = f(p,q)$. This function is *uniformly continuous* in x over K, a subset of M, if $\epsilon > 0$ implies the existence of $\eta > 0$ such that $\delta(x,x_0) < \eta$ implies that $\rho(f(x,y),f(x_0,y)) < \epsilon$ for all y in the subset K of M. If K = M, we say merely that f(x,y) is uniformly continuous in x. f(x,y) is *simultaneously uniformly continuous* in both arguments if $\epsilon > 0$ implies the existence of $\eta_1 > 0$ and $\eta_2 > 0$ such that $\delta(x,x_0) < \eta_1$ and $\delta(y,y_0) < \eta_2$ imply that $\rho(f(x,y),f(x_0,y_0)) < \epsilon$.

In particular, the distance function on any metric space is simultaneously uniformly continuous. This is an immediate consequence of the triangle inequality.

The application of these types of continuity to groupoids which are also metric spaces should be pointed out. The binary operation in a groupoid is a single valued function of two arguments, and therefore, it is meaningful to speak of the operation as continuous in one argument, simultaneously continuous, etc.

Certain algebras are metrized in terms of a real non-negative function, called a norm, defined on each of its elements and satisfying certain conditions. In an additively written group G, this norm, written $|x|$, must satisfy

(N1) $|x| = |-x|$ for all x in G, or an element and its inverse have the same norm;

(N2) $|x| = 0$ if and only if x is the identity 0;

(N3) $|x + y| \leq |x| + |y|$.

The metric on a normed group is a simple generalization of the distance function on the Euclidean line. In the normed group G, $\partial(x,y)$ is defined to be $|x - y|$. The metric space postulates follow directly from (N1) - (N3).

A norm for a lattice L must be

(a) strictly monotone increasing; $x < y$ implies $|x| < |y|$,

(b) modular; $|a| + |b| = |a \vee b| + |a \wedge b|$.

In a normed lattice, the distance function is $\partial(a,b) = |a \vee b| - |a \wedge b|$.

The norm for a real (rational) vector space must be a norm for the additive group of the vector space, i.e., it must satisfy (N1) - (N3) above, and also

(N4) $|\alpha x| = |\alpha||x|$,

where $|\alpha|$ denotes the absolute value of the real (rational) number α. The metric for a vector space is the same as for its additive group.

If a normed real vector space is metrically complete, it is called a Banach space.

In the sequel, when we speak of a normed group or lattice, it will be assumed that the space has been metrized in the manner described above.

CHAPTER III

ALGEBRAIC PROPERTIES IN METRIC SPACES

Let G(ab) be a groupoid which is also a metric space under the distance function $\partial(a,b)$. We will call G(ab) *pseudo-metric*, *metric*, *uniformly pseudo-metric*, or *uniformly metric*, according as the operation ab is continuous in each argument, simultaneously continuous in both arguments, uniformly continuous in each argument, or simultaneously uniformly continuous in both arguments, respectively. It follows from a theorem of E. H. Moore in general analysis that a uniformly pseudo-metric groupoid is necessarily a metric groupoid.

Under the hypothesis of one or another of these types of continuity, certain algebraic properties of a subset of a metric space are extendable to the closure of that subset. Let S be a subset of a metric space, and $\overline{S}$ its closure.

Theorem 3.1. *If* S *is an ideal of a pseudo-metric groupoid, then* $\overline{S}$ *is also an ideal*.

Proof. Let S be an ideal in the pseudo-metric groupoid G(ab). Select any point x in $\overline{S}$. We may write

x as $\lim\limits_{n\to\infty} x_n$, with each x_n in S, since every point in $\overline{S}$ is a point of S, or the limit of a sequence from S. In the first case, x is the limit of the sequence made up of x in every place. For y any element of G, $x_n y$ and $y x_n$ are in S by the ideal property, and we have from the pseudo-metricity of G that xy and yx, the limits of these two sequences, are in $\overline{S}$.

Theorem 3.2. *If* S *is a subgroupoid of a metric groupoid, then* $\overline{S}$ *is also a subgroupoid.*

Proof. Select x and y in $\overline{S}$. Then, as in Theorem 3.1, $x = \lim\limits_{n\to\infty} x_n$, and $y = \lim\limits_{n\to\infty} y_n$, where x_n and y_n are in S for all n. Since $xy = \lim\limits_{n\to\infty} x_n y_n$ by the simultaneous continuity, xy must be in $\overline{S}$.

Theorem 3.3. *If* e *is an identity for a subset* S *of the pseudo-metric groupoid* G, *then* e *is an identity for* $\overline{S}$.

Proof. Consider xe for x any element in $\overline{S}$. As before, $x = \lim\limits_{n\to\infty} x_n$, where each x_n is in S. $xe = \lim\limits_{n\to\infty} x_n e$ $= \lim\limits_{n\to\infty} x_n = x$.

Theorem 3.4. *If the commutative law holds over a subset* S *of a metric groupoid, then the commutative law holds over* $\overline{S}$.

Proof. Select x and y in $\overline{S}$, where $x = \lim\limits_{n\to\infty} x_n$, and $y = \lim\limits_{n\to\infty} y_n$. Then $xy = \lim\limits_{n\to\infty} x_n y_n = \lim\limits_{n\to\infty} y_n x_n = yx$ by the simultaneous continuity and commutativity of S.

Theorem 3.5. If the associative law holds over a subset S of a metric groupoid, then the associative law holds over $\overline{S}$.

Proof. Select x, y, and z in $\overline{S}$, where we may write $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$, and $z = \lim_{n\to\infty} z_n$. Then $xy = \lim_{n\to\infty} x_n y_n$ and $yz = \lim_{n\to\infty} y_n z_n$. However, $x_n(y_n z_n) = (x_n y_n)z_n$ for all n, since these elements are all in S, which was assumed to be associative. Since limits of sequences are unique in metric spaces, it follows that $x(yz) = (xy)z$.

CHAPTER IV

COMPLETION OF UNIFORMLY METRIC GROUPOIDS

Throughout this section, G(ab) denotes a uniformly metric groupoid, and C denotes the Cantor-Hausdorff completion of the metric space G. We will write an element of C as $\{(x_n)\}$ where (x_n) is a representative of the equivalence class of Cauchy sequences in G which constitute the element of C. It should be noted again that C is complete, and that the mapping $x \rightarrow \{(x)\}$, where (x) represents the sequence $x, x, \ldots$, is an isometry mapping sending G into a dense subset of C, which we shall denote by G'.

Lemma 4.1. *If we define an operation* Δ *in* C *in this manner:* $\{(x_n)\} \Delta \{(y_n)\} = \{(x_ny_n)\}$, *then* C *forms a groupoid under this operation, and the isometry mapping of* G *onto* G' *is an isomorphism*.

Proof. An isometry is necessarily one-to-one, or it would not preserve distances. If C is a groupoid, it will be obvious that this mapping is an isomorphism from the definition of the operation in C. To show that C is a groupoid, we must show that the product of any two elements of C is actually in C, and that this product is uniquely determined independent of the representatives chosen from the equivalence classes to perform the operation indicated on the right side of the equality. That the product is in C follows from the uniform metricity of G(ab). $\partial(x_n y_n, x_m y_m) \leq \partial(x_n y_n, x_n y_m) + \partial(x_n y_m, x_m y_m)$, so that if (x_n) and (y_n) are Cauchy sequences, the two terms on the right side of this inequality can be made arbitrarily small. Then $(x_n y_n)$ is a Cauchy sequence and is in some class in C. To show the independence of choice of representatives, let (x_n') and (y_n') be from the same classes as (x_n) and (y_n) respectively. Then, since they are in the same class, for $\eta > 0$, there is an integer N such that $n > N$ implies $\partial(x_n, x_n') < \eta$. Similarly, for $n > M$, $\partial(y_n, y_n') < \eta$. Let K be the larger of M and N. Select an arbitrary $\epsilon > 0$. From the uniform metricity of G(ab) there is $\eta > 0$ such that $\partial(x, x') < \eta$ and $\partial(y, y') < \eta$ imply $\partial(xy, x'y') < \epsilon$. Combining these statements, we have that $\epsilon > 0$ implies the existence of K such that $n > K$ implies that $\partial(x_n y_n, x_n' y_n') < \epsilon$. This is precisely the requirement that $(x_n y_n)$ and $(x_n' y_n')$ be in the same equivalence class, and hence the operation in C is independent of representatives.

Lemma 4.2. *With multiplication defined as in Lemma* 4.1, C *is uniformly metric*.

Proof. For an arbitrary $\epsilon > 0$, there are $\eta_1 > 0$ and $\eta_2 > 0$ such that $\partial(x,x') < \eta_1$ and $\partial(y,y') < \eta_2$ imply that $\partial(xy,x'y') < \epsilon$. Suppose $\rho(\{(x_n)\},\{(x'_n)\}) = \lim_{n\to\infty} \partial(x_n,x'_n) < \eta_1$ and $\rho(\{(y_n)\},\{(y'_n)\}) < \eta_2$. Then there is an N such that $n > N$ implies that $\partial(x_n,x'_n) < \eta_1$ and $\partial(y_n,y'_n) < \eta_2$, since otherwise the limits of these distances could not be less than η_1 and η_2, respectively. Hence, for $n > N$, $\partial(x_n y_n, x'_n y'_n) < \epsilon$ and $\rho(\{(x_n)\}\Delta\{(y_n)\},\{(x'_n)\}\Delta\{(y'_n)\}) = \rho(\{(x_n y_n)\},\{(x'_n y'_n)\}) = \lim_{n\to} \partial(x_n y_n, x'_n y'_n) \leq \epsilon$. This last is precisely the requirement for uniform metricity.

Theorem 4.1. *If* G(ab) *is a uniformly metric groupoid,* G(ab) *may be isomorphically isometrically embedded in a complete metric groupoid* C (*the Cantor-Hausdorff completion*) *in which the image of* G(ab) *is dense*. *If* G *has an identity, then* C *will have an identity*. C *will be commutative, or be associative, if and only if* G *is commutative, or is associative, respectively*. *Moreover, the uniform metricity property will be carried over to* C.

Proof. This follows from Lemmas 4.1 and 4.2, and Theorems 3.3, 3.4, and 3.5 when we note that the Cantor-Hausdorff completion of a metric space is a closure of the space (more precisely, an isometric image of the space.)

CHAPTER V

COMPLETION OF A UNIFORMLY METRIC GROUP

A topological group is a group in which both the binary composition and the operation of taking inverses (inversion) are continuous.

Theorem 5.1. *If* $G(ab)$ *is a uniformly pseudo-metric group, then inversion is uniformly continuous in* G *and* G *is a topological group.*

Proof. For an arbitrary $\epsilon > 0$, there is $\eta > 0$ such that $\partial(a,a') < \eta$ implies $\partial(ab,a'b) < \epsilon$ for all b, from the uniform continuity in each argument. Similarly, given $\eta > 0$, there is $\delta > 0$ such that $\partial(b,b') < \delta$ implies $\partial(ab,ab') < \eta$ for all a. Let $\partial(x,x_0) < \delta$. Then $\partial(x^{-1}x,x^{-1}x_0) = \partial(e,x^{-1}x_0) < \eta$, and

$$\partial(ex_0^{-1},(x^{-1}x_0)x_0^{-1}) = \partial(x_0^{-1},x^{-1}(x_0x_0^{-1})) = \partial(x_0^{-1},x^{-1}) < \epsilon.$$

This is the condition that inversion be uniformly continuous. That xy is continuous follows from the Moore Theorem previously mentioned.

Corollary. If (x_n) *is a Cauchy sequence in a uniformly metric group, then* (x_n^{-1}) *is a Cauchy sequence.*

Theorem 5.2. *Any uniformly metric group is a topological group and may be isometrically isomorphically embedded in a complete uniformly metric topological group in which its image is dense. The completion will be commutative if and only if the original group is commutative.*

Proof. This follows from Theorems 4.1. and 5.1. and the corollary. The corollary assures us that the inverse of an element in the completion is also in the completion. The closure, associativity, identity, and commutativity properties follow from Theorem 4.1.

We note that every normed commutative group, under the metric $\delta(x,y) = |x - y|$, is a uniformly metric group (6).

CHAPTER VI

EMBEDDING A GROUP IN A BANACH SPACE

In this chapter we shall use the additive notation for the normed group G. mx, where m is any non-zero integer, means $x + x + \dots + x$, the "sum" of m x's associated in any way, if m is positive, and $(-x) + (-x) + \dots + (-x)$ if m is negative. $0x$ gives the group identity for all x. We note that $(m + n)x = mx + nx$, $m(nx) = (mn)x$, where m and n are any integers and x is any element of G. Also, since G will be assumed to be commutative, we can "add" in any order: $n(x + y) = nx + ny$ for all integers n and for all x and y in G.

Now let G be an additively written normed commutative group. In addition to the norm requirements for a group, the norm on G is assumed to satisfy

(P) $|mx| = |m||x|$,

where m is any integer, and x any element of G. We construct a new set $B = I \times G$, the set of all pairs (m,a), where m is any non-zero integer and a is any element in the group G. In this set we define

(A) equality: $(n,a) = (m,b)$ if and only if $nb = ma$.

(B) addition: $(n,a) + (m,b) = (mn, ma + nb)$;

(C) a norm: $|(n,a)| = \left|\frac{1}{n}\right| |a|$.

This construction has also been employed by Neumann (21) under different hypotheses.

From (A), $(n,a) = (n,a)$, and if $(n,a) = (m,b)$, then $(m,b) = (n,a)$. If $(n,a) = (m,b)$ and $(m,b) = (q,c)$, then $nb = ma$ and $mc = bq$. Multiply the first equation by q and the second by n. Then $qma = qnb = mnc$. Adding $-mnc$ to both sides, we get that $m(qa - nc) = 0$. But this cannot be true unless $qa - nc = 0$, since otherwise (P) would not be satisfied. Hence, $(n,a) = (q,c)$, and (A) is an equivalence relation.

We note that $(n,0) = (m,0)$ for all n and m, and this element is the group identity, since $(m,a) + (n,0) = (mn, na) = (m,a)$. The inverse of (n,a) is $(-n,a)$, since $(n,a) + (-n,a) = (-n^2,0)$. Repeated addition follows the rule

(D) $m(n,a) = (n, ma)$.

Consider the mapping $f(a) = (1,a)$ of G into B. $f(a + b) = (1, a + b) = (1,a) + (1,b)$; hence, since it is clearly biuniform, f is an isomorphism. Further, the norms of the elements of this subset of B are the same as the norms of their originals in G. Hence, distances are preserved by this mapping, and we have

Theorem 6.1. *The extension B is a commutative group which is normed by (C), and which contains a subgroup isomorphic and isometric to G under the mapping* $f(a) = (1,a)$.

If we define $\frac{p}{q}(n,a)$ to be $p(qn,a)$, (D) gives the rule for finding the product of an element of B by the rational number $\frac{p}{q}$. Let α and β be the rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, respectively. Then, using (A), (B), and (C):

$$\begin{aligned}(\alpha + \beta)(n,a) &= (\tfrac{p}{q} + \tfrac{r}{s})(n,a) \\ &= (\tfrac{ps + qr}{qs})(n,a) \\ &= (qsn,(ps + qr)a) \\ &= (qsn^2,psna + qrna) \\ &= (qn,pa) + (sn,ra) \\ &= \tfrac{p}{q}(n,a) + \tfrac{r}{s}(n,a) \\ &= \alpha(n,a) + \beta(n,a);\end{aligned}$$

Similarly, $\alpha((n,a) + (m,b)) = \alpha(n,a) + \alpha(m,b)$, and $\alpha(\beta(n,a)) = (\alpha\beta)(n,a)$. Also, $1(n,a) = (n,a)$. Hence, we have embedded G in a normed rational vector space, the norm being given by (C). We express this in

Theorem 6.2. *The extension B is a normed rational vector space whose additive group contains a subgroup isometrically isomorphic to G.*

Theorem 6.3. The Cantor-Hausdorff completion of a normed rational vector space is a real vector space which can be normed, and hence is a Banach space.

Proof. If R is a normed rational vector space, it is a uniformly metric group, and its completion C is also a uniformly metric group by Theorem 5.2. C will also be commutative.

Let X denote the Cauchy sequence equivalence class $\{(x_n)\}$ in the completion C. Define $|X| = \lim_{n\to\infty} |x_n|$. This limit exists since $|x|$ is a continuous function of x. Then $|X - Y| = |\{(x_n - y_n)\}| = \lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} ə(x_n, y_n) = \rho(X,Y)$, the metric in C. Thus, C is a normed commutative group.

Define λX for λ rational to be the class $\{(\lambda x_n)\}$. (This a Cauchy sequence class, since $ə(\lambda x_i, \lambda x_j) = |\lambda x_i - \lambda x_j| = |\lambda(x_i - x_j)| = |\lambda||x_i - x_j| = \lambda ə(x_i, x_j) < \epsilon$ for $ə(x_i, x_j) < \frac{\epsilon}{\lambda}$.) Then C admits operators from the rationals and

$$(\lambda + \mu)X = \{((\lambda + \mu)x_n)\} = \{(\lambda x_n + \mu x_n) = \lambda X + \mu X;$$

$$\lambda(X + Y) = \{(\lambda(x_n + y_n))\} = \{(\lambda x_n + \lambda y_n) = \lambda X + \lambda Y;$$

$$\lambda(\mu X) = \{(\lambda(\mu x_n))\} = \{((\lambda\mu)x_n)\} = (\lambda\mu)X;$$

$$1X = (1x_n) = (x_n) = X.$$

Further, $|\lambda X| = \lim_{n\to\infty} |\lambda x_n| = |\lambda| \lim_{n\to\infty} |x_n| = |\lambda||X|$.

Choose a sequence of rationals (λ_n) which approaches α, a real number. Form the sequence $\lambda_1 X$, $\lambda_2 X, \ldots$ of elements of C. This is a Cauchy sequence, since $\rho(\lambda_i X, \lambda_j X) = |\lambda_i X - \lambda_j X| = |(\lambda_i - \lambda_j)X| = |\lambda_i - \lambda_j||X|$, where $|X|$ is finite and $|\lambda_i - \lambda_j|$ can be made arbitrarily small. This Cauchy sequence in the completion must have a limit. Define αX to be this limit. Thus, multiplication by the reals is defined in C. Let $\alpha = \lim_{i\to\infty} \lambda_i$, $\beta = \lim_{j\to\infty} \mu_j$. Then

$$(\alpha + \beta)X = \lim_{i\to\infty} (\lambda_i + \mu_i)X = \lim_{i\to\infty} \lambda_i X + \lim_{i\to\infty} \mu_i X$$
$$= \quad X + \quad X;$$

$$\alpha(X + Y) = \lim_{i\to\infty} \lambda_i(X + Y) = \lim_{i\to\infty} \lambda_i X + \lim_{i\to\infty} \lambda_i Y$$
$$= \quad X + \quad Y;$$

$$\alpha(\beta X) = \lim_{i\to\infty} \lambda_i(\beta X) = \lim_{i\to\infty} \lambda_i \lim_{j\to\infty} \mu_j X$$
$$= \lim_{i\to\infty} \lim_{j\to\infty} \lambda_i \mu_j X = (\alpha\beta)X;$$

and as before, $1X = X$.

By the proof of the Cantor-Hausdorff completion theorem, αX is the limit of the sequence $(\lambda_k x_{m_k})$, where (x_{m_k}) is a subsequence of the Cauchy sequence (x_n) in the equivalence class denoted by X. Then, from the definition of the norm in C,

$$|\alpha X| = \lim_{k\to\infty} |\lambda_k x_{m_k}| = \lim_{k\to\infty} |\lambda_k||x_{m_k}| = |\alpha||X|:$$

Thus, C is shown to be a complete normed real vector space, and hence a Banach space.

Remembering that G was isometrically isomorphically embedded in B, and that B is isometrically isomorphically embedded in C, we have

Theorem 6.4. *Any normed commutative group satisfying (P) can be embedded in a Banach space.*

CHAPTER VII

METRIC Δ-LATTICES

Several studies (1,4,9,11,13) have been concerned with lattices which form a groupoid under a third operation. Under different hypotheses as to the relation that this third operation bears to the two lattice operations, various systems of interest result. The structure we wish to investigate is similar to what Birkhoff (4) calls a commutative m-lattice. Somewhat related systems are the lattice ordered semigroup and the lattice ordered group.

Consider now a lattice L in which a third binary operation, Δ-multiplication, is defined under which L forms a groupoid. We further require

(A) Commutativity: $a \Delta b = b \Delta a$, for all a and b in L;

(B) Distributivity of Δ over meet and join: $a \Delta (b \wedge c) = (a \Delta b) \wedge (a \Delta c)$ and $a \Delta (b \vee c) = (a \Delta b) \vee (a \Delta c)$, for all a, b, and c in L.

A lattice with Δ and (A) and (B) is called a Δ-lattice. If L is a normed Δ-lattice in which Δ-translations are isometries, then L is a metric Δ-lattice. If, further,

Δ-multiplication is associative, L is an associative metric Δ-lattice.

An interesting example of an associative metric Δ-lattice is obtained as follows: let E_n be the n-dimensional Euclidean space. Take the convex body defined by the hyperplanes $\sum_1^n \epsilon_i x_i = 1$, where $\epsilon_i = \pm 1$. The distance function in the non-negative quadrant of the n-dimensional Minkowski space defined by this convex body will be the distance function of a normed lattice formed on this quadrant by the ordering $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if $x_i \leq y_i$ for all i, and the norm $|(x_1, x_2, \ldots, x_n)| = \sum_1^n x_i$. This normed lattice with Δ-multiplication defined as vector addition forms an associative metric Δ-lattice.

Let α and β be any two mappings of a normed lattice into itself. Define a mapping $\alpha V \beta$ by $x(\alpha V \beta) = x\alpha V x\beta$, and $\alpha \wedge \beta$ by $x(\alpha \wedge \beta) = x\alpha \wedge x\beta$.

Now let E be a set of isometric endomorphisms on a normed lattice L. If E is closed under $\alpha \wedge \beta$ and $\alpha V \beta$ as defined above, it follows from the lattice properties of L that E is a lattice. If E is also closed under endomorphism multiplication, E is called an isometric endomorphism lattice. If endomorphism multiplication commutes, E is a commutative isometric endomorphism lattice.

It should be noted that, from (B), Δ-translations in a Δ-lattice are lattice endomorphisms, and thereby preserve order.

Theorem 7.1. *In a Δ-lattice,* $a \Delta b = (a \wedge b) \Delta (a \vee b)$ *for all* a *and* b *in the lattice.*

Proof. $(a \wedge b) \Delta (a \vee b)$

$$= ((a \wedge b) \Delta a) \vee ((a \wedge b) \Delta b)$$
$$= ((a \Delta a) \wedge (a \Delta b)) \vee ((a \Delta b) \wedge (b \Delta b))$$
$$\leq (a \Delta b) \vee ((a \Delta b) \wedge (b \Delta b))$$
$$= a \Delta b.$$

Also, $(a \vee b) \Delta (a \wedge b)$

$$= ((a \vee b) \Delta a) \wedge ((a \vee b) \Delta b)$$
$$= ((a \Delta a) \vee (b \Delta a)) \wedge ((a \Delta b) \vee (b \Delta b))$$
$$> ((a \Delta a) \vee (b \Delta a)) \wedge (a \Delta b)$$
$$= a \Delta b.$$

By the antisymmetry of $\leq$, $(a \wedge b) \Delta (a \vee b) = a \Delta b$.

This relation between the three operations in the lattice has been demonstrated by Dedekind (9), Freudenthal (13), Birkhoff (4), Everett (11), and Arnold (1), all under more restrictive hypotheses. Since it clearly implies the uniqueness of relative complements, a Δ-lattice is necessarily a distributive lattice. In this same connection, it should be noted that a metric Δ-lattice satisfies the cancellation law for Δ-products, since isometries are biuniform.

We shall need the following result of Blumenthal and Ellis (5):

Lemma 7.1. *If* f *is a one-to-one mapping of a normed lattice* L *onto a normed lattice* L' *which possesses any two of the following properties,*

(1) f *preserves norms modulo a constant*;

(2) f *preserves meets*;

(3) f *preserves distances*;

then f *also possesses the third property*.

Corollary. *If* α *is an isometric lattice endomorphism on the normed lattice* L, α *preserves norms modulo a non-negative constant*. *If, moreover, the norms in* L *are bounded, this constant is* 0.

Proof. By the Archimedean property of the real field, there exists an integer n such that $n|k| > a$, where k is the constant guaranteed by Lemma 7.1, and a is an arbitrary but fixed element of L. Now, $|a\alpha| = |a| + k$, $|a\alpha^2| = |a\alpha| + k = |a| + k + k$. Similarly, $|a\alpha^n| = |a| + nk$. If k were negative, $|a\alpha^n|$ would be negative, which is impossible. If the norms are bounded, k must be 0, since otherwise, if M is the bound, there would exist an n such that $nk > M$, and hence $|a\alpha^n| = |a| + nk$ would be $> M$.

Lemma 7.1 and the corollary enable us to associate with each isometric endomorphism of a normed lattice a non-negative constant. If α is an isometric endomorphism on the normed lattice L, define $|\alpha|$ by the equality $|x\alpha| = |x| + |\alpha|$. In particular, in a metric Δ-lattice, for each element a there is a non-negative constant λ_a such that $|a \Delta b| = |b| + \lambda_a$. At the same time, $|a \Delta b| = |a| + \lambda_b$, or $|a| - |b| = \lambda_a - \lambda_b$. In the operator notation, where R_a denotes the translation by a, we have $|xR_a| = |x| + \lambda_a$, so that $|R_a| = \lambda_a$. Also from Lemma 7.1, we can make an equivalent definition of a metric Δ-lattice. A metric Δ-lattice is a normed Δ-lattice in which $|a \Delta x| = |x| + \lambda_a$ for all a and x, where λ_a is a non-negative function of a. For, Δ-translations preserve meets by (B), and preserve norms modulo a constant by definition. Then, by Lemma 7.1, they are isometries.

Lemma 7.2. *Every commutative isometric endomorphism lattice on a normed lattice L is an associative metric Δ-lattice.*

Proof. Let A be a commutative isometric endomorphism lattice on the normed lattice L. For α and β in A, $|\alpha \vee \beta| + |\alpha \wedge \beta|$

$$= |x(\alpha \vee \beta)| - |x| + |x(\alpha \wedge \beta)| - |x|$$

$$= |x\alpha \vee x\beta| - |x| + |x\alpha \wedge x\beta| - |x|$$

$$= |x\alpha| + |x\beta| - |x\alpha \wedge x\beta| - |x| + |x\alpha \wedge x\beta| - |x|$$
$$= |x\alpha| + |x\beta| - 2|x|$$
$$= |x| + |\alpha| + |x| + |\beta| - 2|x|$$
$$= |\alpha| + |\beta|.$$

If $\alpha < \beta$, $x\alpha < x\beta$, since $\alpha \wedge \beta = \alpha$, and $\alpha \neq \beta$. Then $|x\alpha| < |x\beta|$. and $|\alpha| = |x\alpha| - |x| < |x\beta| - |x| = |\beta|$. Thus A is a normed lattice. Endomorphism multiplication commutes by hypothesis. $x(\alpha \vee \beta)\eta = (x\alpha \vee x\beta)\eta = (x\alpha\eta \vee x\beta\eta) = x(\alpha\eta \vee \beta\eta)$, so that multiplication distributes over join, and similarly over meet. Also

$$|\alpha\beta| = |x(\alpha\beta)| - |x|$$
$$= |(x\alpha)\beta| - |x|$$
$$= |x\alpha| + |\beta| - |x|$$
$$= |x| + |\alpha| + |\beta| - |x|$$
$$= |\alpha| + |\beta|.$$

Hence, we have satisfied the second definition of a metric Δ-lattice. We have here the special case in which $\lambda_\alpha = |\alpha|$. Now, $x(\alpha\beta)\gamma = ((x\alpha)\beta)\gamma = (x\alpha)\beta\gamma$, and therefore $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, the associative law.

In the sequel, we shall always understand that a commutative isometric endomorphism lattice is normed in the above fashion.

Theorem 7.2. *If L is an associative metric Δ-lattice,* *then it is isometrically lattice isomorphic to a commutative isometric endomorphism lattice on L composed of the Δ-translations of L. Moreover, this lattice isomorphism is a multiplicative isomorphism.*

Proof. Let S be the set of all Δ-translations on L. These form a lattice under $R_a \vee R_b$ and $R_a \wedge R_b$ as defined for two arbitrary mappings, since $R_a \quad R_b = R_{a\vee b}$, and $R_a \wedge R_b = R_{a\wedge b}$. Further, $R_aR_b = R_bR_a$ since $x(R_aR_b) = (xR_a)R_b = (x \Delta a) \Delta b = x \Delta (a \Delta b) = xR_{a\Delta b} = xR_{b\Delta a} = x(R_bR_a)$. Remembering that Δ-translations are isometries, it is now clear that S is a commutative isometric endomorphism lattice. The mapping $a \rightarrow R_a$ is clearly a lattice isomorphism. Also

$$\begin{aligned} \partial(R_a,R_b) &= |R_a \vee R_b| - |R_a \wedge R_b| \\ &= \lambda_{a\vee b} - \lambda_{a\wedge b} \\ &= |a \vee b| - |a \wedge b| \\ &= \partial(a,b), \end{aligned}$$

and thus, it is also an isometry. $a \Delta b \rightarrow R_{a\Delta b} = R_aR_b$, which establishes that it is also a multiplicative isomorphism.

Theorem 7.3. *If L is a normed lattice which is isometrically lattice isomorphic to some commutative isometric endomorphism lattice S on L, then a multiplication may be introduced in L under which L forms an associative metric Δ-lattice, and the given isometric lattice iso-*

morphism of L onto S will also be a multiplicative isomorphism.

Proof. By Lemma 7.2, S is an associative metric Δ-lattice in which $|\alpha\beta| = |\alpha| + |\beta|$. To define a multiplication in L, let f be the isometric lattice isomorphism of L onto S. For a and b in L, define $a \Delta b = f^{-1}(f(a)f(b))$, where f^{-1} is the inverse of the mapping f. By Lemma 7.1, f preserves norms modulo a constant, so that $|a| = |f(a)| + k$. Now,

$$\begin{aligned} a \Delta (a \vee c) &= f^{-1}(f(a)f(b \vee c)) \\ &= f^{-1}(f(a)(f(b) \vee f(c))) \\ &= f^{-1}(f(a)f(b) \vee f(a)f(c)) \\ &= f^{-1}(f(a)f(b)) \vee f^{-1}(f(a)f(c)) \\ &= (a \Delta b) \vee (a \Delta c), \end{aligned}$$

so that Δ-multiplication distributes over join, and similarly over meet. The commutativity and associativity of Δ and the fact that f is also a multiplicative isomorphism follow directly from the definition of $a \Delta b$. It remains to show that translations in L are isometries.

$$\begin{aligned} |a \Delta b| &= |f(f^{-1}(f(a)f(b)))| + k \\ &= |f(a)f(b)| + k \\ &= |f(a)| + |f(b)| + k \\ &= |a| - k + |b| - k + k \\ &= |a| + |b| - k. \end{aligned}$$

Then, $\partial(a \Delta c, b \Delta c)$

$$\begin{aligned} &= |(a \Delta c) \vee (b \Delta c)| - |(a \Delta c) \wedge (b \Delta c)| \\ &= |(a \vee b) \Delta c| - |(a \wedge b) \Delta c| \\ &= |a \vee b| + |c| - k - |a \wedge b| - |c| + k \\ &= |a \vee b| - |a \wedge b| \\ &= \partial(a,b). \end{aligned}$$

Thus, the multiplication defined in L makes L an associative metric Δ-lattice, and the theorem is proved.

Combining Theorems 7.2 and 7.3, we get the following characterization of associative metric Δ-lattices among metric lattices:

Theorem 7.4. *Let* L *be a normed lattice*. *Then* L *admits a multiplication to form an associative metric* Δ-*lattice if and only if* L *is isometrically lattice isomorphic to a commutative isometric endomorphism lattice on itself, and in this case the isometric lattice isomorphism between* L *and the operator lattice is also a multiplicative isomorphism*.

Wilcox and Smiley (23) have shown that any normed lattice can be embedded isomorphically and isometrically in a metrically complete lattice. Since Δ-translations are isometries, it is easily seen that the Δ-operation is simultaneously uniformly continuous, so that it is possible to define a Δ-multiplication in the completion. This operation will distribute over meet and join, and Δ-translations will be isometries. Hence, we have that an associative metric Δ-lattice can be isometrically isomorphically embedded in a metrically complete associative metric Δ-lattice.

CHAPTER VIII

GAUSSIAN SEMIGROUPS

In this chapter, we investigate the relation that a suitably restricted associative metric Δ-lattice bears to a reduced Gaussian semigroup. We shall need the equivalent definition of a metric Δ-lattice furnished by

Theorem 8.1. A metric Δ-lattice is precisely a Δ-lattice on which is defined a non-negative strictly monotone increasing real function $|x|$ such that $|a \Delta b| = |a| + |b| + k$, where k is a constant.

Proof. Let L be a metric Δ-lattice. Let $|x|$ be the norm of L. Then $|a \Delta b| = |a| + \lambda_b = |b| + \lambda_a$. Let $k = \lambda_b - |b|$. Then k is independent of b since $\lambda_b - |b| = \lambda_a - |a|$, and we have $|a \Delta b| = |a| + |b| + k$. Conversely, suppose L is a Δ-lattice and $|x|$ is a non-negative strictly monotone increasing real function on L with $|a \Delta b| = |a| + |b| + k$. Now, by Theorem 7.1, $a \Delta b = (a \vee b) \Delta (a \wedge b)$, so $|a| + |b| = |a \Delta b| - k = |(a \vee b) \Delta (a \wedge b)| - k = |a \vee b| + |a \wedge b|$, and the function is a lattice norm. To show that Δ-translations are isometries, we observe

$$\begin{aligned} \partial(a \Delta b, a \Delta c) &= |(a \Delta b) \vee (a \Delta c)| - |(a \Delta b) \wedge (a \Delta c)| \\ &= |a \Delta (b \vee c)| - |a \Delta (b \wedge c)| \\ &= |a| + |b \vee c| + k - (|a| + |b \wedge c| + k) \\ &= |b \vee c| - |b \wedge c| \\ &= \partial(b,c). \end{aligned}$$

Theorem 8.2. Any reduced Gaussian semigroup may be made into an associative metric Δ-lattice with lattice ordering as divisibility.

Proof. Let $G(ab)$ be a reduced Gaussian semigroup, and let S be the set of irreducibles of G. Let f be any function which assigns a positive real number to each element of S. Set $|e| = 0$, where e is the identity of $G(ab)$, and let $|p| = f(p)$ for p an element of S. Any element of G has a unique factorization into irreducibles. If a has the factorization $a = \prod_i p_i^{\alpha_i}$, define $|a| = \sum_i \alpha_i |p_i|$. Define $a \leq b$ if $a|b$. This ordering makes G a partially ordered set; and since the g.c.d. of two elements serves as their meet, and the l.c.m. as their join, G is, in fact, a lattice. Moreover, $a(b \vee c) =$ a(l.c.m of b and c) $=$ l.c.m of ab and $ac = ab \vee ac$, and similarly multiplication distributes over meet. The commutativity and associativity of multiplication follow from the definition of a Gaussian semigroup. It is clear from the definition of $|x|$ that $|ab| = |a| + |b|$, so that by Theorem 8.1, it suffices to show that $|x|$ is strictly monotone increasing. If $a|b$ and $a \neq b$, then there is an element x in G, $x \neq e$, so that $ax = b$. Hence $|b| = |a| + |x| > |a|$. Thus the theorem is proved.

We call a Δ-lattice *transitive above* if, for $x \leq y$, there is z so that $x \Delta z = y$. A normed lattice with first element is called *atomically bounded* if it is an atomic lattice in which the norms of the atoms preceding a given element are bounded away from 0. We shall assume that if a normed lattice has a first element, the norms have been reduced modulo a constant so that the norm of the first element is 0.

Lemma 8.1. *An atomic associative metric Δ-lattice which is transitive above has a multiplicative identity which is the first element of the lattice*.

Proof. Let L be an associative metric Δ-lattice which is transitive above. For each x in L there is e_x with $x \Delta e_x = x$. Similarly, $y \Delta e_y = y$, so that $x \Delta y \Delta e_x = x \Delta y \Delta e_y$, and, by cancellation, $e_x = e_y$. Thus, there is an identity element e. Since $|x \Delta e| = |x| + \lambda_e$, $\lambda_e = 0$. Then $|x| - |e| = \lambda_x$, which is non-negative, and thus $|e| \leq |x|$ for all x. But since L is a normed lattice, the first element, and only the first element, has norm 0, and hence e is the first element.

Lemma 8.2. *The product of distinct atoms in an atomic associative metric Δ-lattice is their join.*

Proof. This follows by induction from Theorem 7.1. If p and q are distinct atoms, $p \Delta q = (p \vee q) \Delta (p \wedge q)$, and since $p \wedge q = e$, we have $p \Delta q = p \vee q$.

It is now clear that in an atomic normed lattice, atomic boundedness is equivalent to the property that a given element is preceded by only a finite number of atoms; for, if there were an infinite number whose norms were not bounded away from 0, the norm of the element which these atoms precede would not be defined, since the element itself has to follow the join of these atoms.

Theorem 8.3. *An atomically bounded associative metric Δ-lattice which is transitive above forms a reduced Gaussian semigroup in which the lattice ordering is divisibility, and the atoms are the irreducibles.*

Proof. The multiplication is associative and commutative from the definition of an associative metric Δ-lattice, and, as we have already noted, the cancellation law is valid. By Lemma 8.1, there is a multiplicative identity e which is the first element of the lattice. Suppose $x \leq y$. Then, by the transitivity above,

$x|y$. Now, $a \wedge (a \,\Delta\, b) = (a \,\Delta\, e) \wedge (a \,\Delta\, b) = a \,\Delta\, (e \wedge b) = a \,\Delta\, b$; therefore, $a \leq a \,\Delta\, b$. Thus, $a|b$ implies $a \leq b$, and the lattice ordering is divisibility. Let A_x be the set of atoms preceding the element x. A_x is finite because the lattice is atomically bounded. Then, by Lemma 8.2, the product of all the atoms in A_x, written $\underset{A_x}{\pi}\, p$, is their join. Since the lattice is transitive above and this product precedes x, there is an element x_1 such that $x = x_1 \underset{A_x}{\pi}\, p$. Now, $x_1|x$, so that the set of atoms preceding x_1 is a subset of A_x. Denoting this set by A_{x_1}, and repeating the process for x_1, there is an element x_2 such that $x = x_2 \underset{A_{x_1}}{\pi}\, p \underset{A_x}{\pi}\, p$. Since the norms of atoms are positive, this reduction must terminate with an $x_n = e$, and x has been decomposed into a product of powers of the atoms preceding it. It remains only to show that this decomposition is unique. To do this, it suffices to show that if an atom divides a product, it divides one of the factors. Then it will be possible to cancel from the two factorizations step by step. Suppose, then, that p is an atom, and $p|ab$, but that p does not divide either a or b. Then $p \wedge a = p \wedge b = e$, or $p \wedge (a \vee b) = e$, since L is distributive. Case I: $a \wedge b \neq e$. Then e, p, $a \vee b$, $a \wedge b$, and $p \vee (a \vee b)$ form a non-modular sublattice of L, which is impossible.

Case II: $a \wedge b = e$. Then $a \vee b = a \Delta b$, and since $p \wedge a$ is assumed to be e, $p \Delta a = p \vee a$. But $p \leqslant a \Delta b$, and $a \leqslant a \Delta b$. By the transitivity above, $p \Delta a \Delta x = a \Delta b$ for some x, but this implies $p|b$, contrary to assumption. Thus, the theorem is proved.

Combining Theorems 8.2 and 8.3, we have

Theorem 8.4. *Algebraically, the class of atomically bounded associative metric* Δ*-lattices which are transitive above is the class of reduced Gaussian semigroups.*

CHAPTER IX

SOME UNSOLVED PROBLEMS

In the development of this dissertation, some problems suggested themselves that, of necessity, had to be left untreated. Some of them are listed here.

1. What are necessary and sufficient conditions that a complete metric groupoid be uniformly metric?

2. In the case of a normed lattice, the relation of modularity between the two operations is carried over into the completion. Also, in the case of a metric Δ-lattice, the distributivity of Δ over join and meet is carried over to the completion. We have dealt in the first part of this dissertation with certain algebraic properties of a single operation which are valid in the metric completion. An interesting study remains: what types of relations among operations in a multiple operation metric algebra carry over to the completion?

3. What are necessary and sufficient conditions on an atomically bounded associative metric Δ-lattice in order that it be transitive above? There is also the same

question for a general metric Δ-lattice.

4. Is there a constructive method for norming a Δ-lattice so that it becomes a metric Δ-lattice?

5. Of course, there still remains the well-known unsolved problem of finding necessary and sufficient conditions in order that a lattice may be normed.

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BIOGRAPHY

Randall Murray Conkling was born in South Glens Falls, New York, on December 9, 1922. He attended the public schools of Glens Falls, New York, through high school, being graduated therefrom in 1939. His undergraduate work was taken at Green Mountain Junior College, Poultney, Vermont; Union College, Schenectady, New York; and Bucknell University, Lewisburg, Pennsylvania, this study being interrupted from 1943 to 1947 for service in the United States Air Force. He received the Bachelor of Science in Education from Bucknell in 1948, and the Master of Science from the same institution in 1949. He was a graduate assistant both at Bucknell and at the University of Florida. He is a member of Pi Mu Epsilon, the Mathematical Association of America, and the American Mathematical Society.

This dissertation was prepared under the direction of the Chairman of the candidate's Supervisory Committee and has been approved by all members of the Committee. It was submitted to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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